

NMSU MATH PROBLEM OF THE WEEK

Solution to Problem 5

Fall 2025

Problem 5

Find the largest number of points which a football team cannot get exactly using just 3-point field goals and 7-point touchdowns (ignore the possibilities of safeties, missed extra points, and two point conversions). Justify your answer.

Solution. Clearly 11 cannot be obtained this way. We show that 11 is the largest such by proving that each $n \geq 12$ can be obtained in this manner. In other words, we prove that each $n \geq 12$ can be written as $n = 3x + 7y$ for some $x, y \geq 0$.

Our proof is by induction. The base case is $n = 12$, which is obvious since $12 = 3 \cdot 4$, so we can take $x = 4$ and $y = 0$. For the induction step, suppose $n \geq 12$ can be written as $n = 3x + 7y$ for some $x, y \geq 0$. We show that the same is true for $n + 1$. We consider cases:

Case 1: Suppose $y = 0$. Then $n = 3x$ and we must have $x \geq 4$. Therefore,

$$n + 1 = 3x + 1 = 3(x - 2) + 3 \cdot 2 + 1 = 3(x - 2) + 7.$$

Case 2: Suppose $y = 1$. Then $n = 3x + 7$ and we must have $x \geq 2$. Therefore,

$$n + 1 = (3x + 7) + 1 = 3(x - 2) + 3 \cdot 2 + 7 + 1 = 3(x - 2) + 7 \cdot 2.$$

Case 3: Suppose $y \geq 2$. Then $n = 3x + 7y$. Therefore, since $y \geq 2$,

$$n + 1 = (3x + 7y) + 1 = 3x + 7(y - 2) + 7 \cdot 2 + 1 = 3x + 3 \cdot 5 + 7(y - 2) = 3(x + 5) + 7(y - 2).$$

Consequently, by mathematical induction, $n = 3x + 7y$ for each $n \geq 12$. Thus, the answer to the problem is 11.